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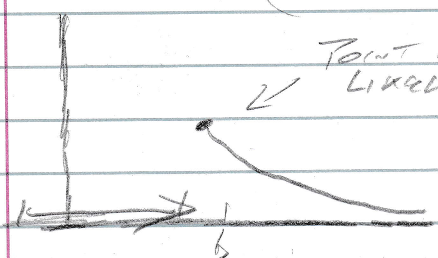
1. $\mathcal{L}(b) = \prod_{i=1}^{n} \frac{1}{b} (\mathbb{1}\{0 \le x_i < b\}$ (CASE OF 0, b)

$= \frac{1}{b^n} \prod \{b > x_i\}$ (TRUE IN ALL CASES)

$\mathbb{1}_{(x_i, \infty)}(b) = \begin{cases} 1 & \text{if } b \ge x_i \\ 0 & \text{OTHERWISE} \end{cases}$

$= \frac{1}{b^n} \mathbb{1}(b)(x_i^{(n)}, \infty)$ TRICKY: DISCONTINUOUS FUNCTION

$= \begin{cases} \frac{1}{b^n} & \text{IF } b \ge x^n \\ 0 & \text{IF } b < x \end{cases}$



$\frac{dy}{db}\, b^{-n} = b^{-n}$

$= -1(b)^{-2}$

$-1(b)^{-2} < 0$

SO THE FUNCTION IS DECREASING

1a) $\mathcal{L}(a,b) = \prod^{n} \frac{1}{b-a} \left(\mathbb{1}\{a < x_i < b\}\right)$ (CASE OF a, b)

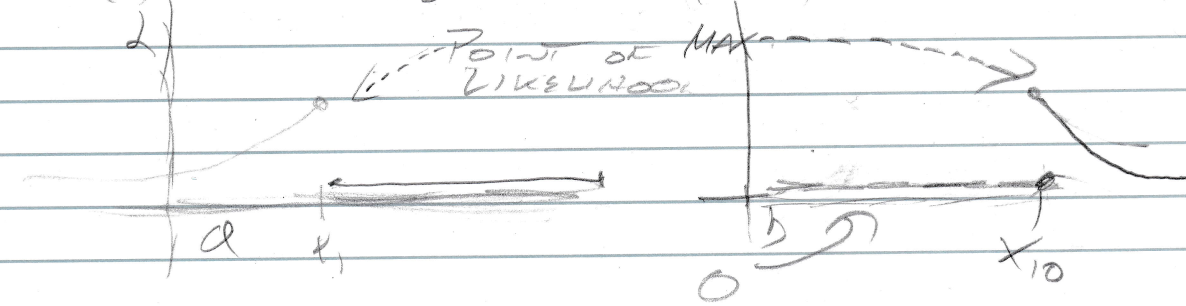
$= \frac{1}{b-a} \prod \{\mathbb{1}_{(-\infty, x_i)}(a)\} \cdot \{\mathbb{1}_{(x_i, +\infty)}(b)\}$

$= \frac{1}{b-a} \mathbb{1}(a)_{(-\infty, x^{(1)})} \cdot \mathbb{1}(b)_{(x^{(10)}, \infty)}$

MAX(a): $\frac{1}{b-a}$ $\frac{d\mathcal{L}}{da} = -1(b-a)^{-2} \cdot (-1) > 0$ (INCREASING)

$\hat{a} = x^{(1)}$

MAX(b): $\frac{1}{b-a}$ $\frac{d\mathcal{L}}{db} \frac{1}{b-a} = -1(b-a)^{-2} \Rightarrow$ (DECREASING)



2. $Y_i = a + bx + \epsilon_i$; $\epsilon_i \sim N(0, \sigma)$

* ASSUMING ϵ_i IS NORMALLY DISTRIBUTED *

$$Y_i \sim N(a + bx_i, \sigma^2)$$

$$\mathcal{L}(a, b, \sigma^2) = \prod_{i=1}^{n} f(y_i)$$

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y_i - (a + bx_i))^2}{\sigma^2}}$$

$$\log \mathcal{L}(a, b, \sigma^2) = \sum \log \frac{1}{\sqrt{2\pi\sigma^2}} + \log(e^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\sum \log(2\pi\sigma^2) - \frac{1}{2}\frac{(y_i - (a + bx_i))^2}{\sigma^2}$$

$$\frac{d \log \mathcal{L}}{da} = 0 + \sum \left(\frac{-1}{\sigma^2}(y_i - (a + bx_i)) * (-1)\right)$$

$$= \sum 1 \frac{(y_i - (a + bx_i))}{\sigma^2} = 0 \quad (\text{MAX})$$

$$\frac{d \log \mathcal{L}}{db} = 0 + \sum \frac{(-1(y_i - (a + bx_i)) * (-x_i)}{\sigma^2}$$

$$= \sum x_i \frac{(y_i - (a + bx_i))}{\sigma^2} = 0 \quad (\text{MAX})$$

$$\frac{d \log \mathcal{L}}{d\sigma} = -\frac{1}{2}\sum \frac{1}{2\pi\sigma^2} * (4\pi\sigma) +$$

$$\left(-\frac{1}{2}\sum (y_i - (a + bx_i))^2 [-2\sigma^{-3}]\right) = 0 \quad (\text{MAX})$$

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$$\begin{cases} \dfrac{\sum (y_i - (a + bx_i))}{\sigma^2} = 0 \\ \dfrac{\sum x_i (y_i - (a + b x_i))}{\sigma^2} = 0 \\ -\frac{1}{2} \sum \frac{1}{2\pi\sigma^2}(4\pi\sigma) - \sum (y_i - (a + bx_i))^2 (-2\sigma^{-3}) = 0 \end{cases}$$

$\sum (y_i - a - bx_i) = 0$

$\sum y_i - \sum a - \sum b x_i = 0 \qquad \sum a = na$

$\sum y_i - na - \sum b x_i = 0$

$\sum y_i - \sum b x_i = na$

$\hat{a} = \dfrac{\sum y_i - \sum b x_i}{n} = \dfrac{\sum y_i}{n} - \dfrac{\sum b x_i}{n}$

$= \bar{Y} - \dfrac{b \sum x_i}{n} = \bar{Y} - \hat{b}\bar{X}$

$\sum x_i (y_i - ((\bar{Y} - b\bar{X}) + bx_i)) = 0$

$\sum x_i y_i - \sum x_i \bar{Y} + b \sum x_i \bar{X} - b \sum x_i^2 = 0$

$\sum x_i y_i - \sum x_i \bar{Y} + b n \bar{x}^2 - b \sum x_i^2 = 0$

$\sum x_i y_i - \sum x_i \bar{Y} + b(n\bar{x}^2 - \sum x_i^2) = 0$

$b(n\bar{x}^2 - \sum x_i^2) = -\sum x_i y_i + \sum x_i \bar{Y}$

$\hat{b} = \dfrac{-\sum x_i y_i + \sum x_i \bar{Y}}{n\bar{x}^2 - \sum x_i^2} = \dfrac{\sum x_i (y_i - \bar{y})}{\sum x_i - n\bar{x}^2}$

$-\frac{1}{2} \sum \frac{1}{2\pi\sigma^2}(4\pi\sigma) - \sum (y_i - (\hat{a} + \hat{b}x_i))^2 (-2\sigma^{-3}) = 0$

$\frac{-n}{\sigma} + \frac{2}{\sigma^3}(\sum e_i^2) = 0$

$\sigma^3\left(\frac{-n}{\sigma} + \frac{2}{\sigma^3}(\sum e_i^2)\right) = -n\sigma^2 + \sum e_i^2 = 0$

$n\sigma^2 = 2\sum e_i^2 \qquad \sigma = \sqrt{\dfrac{\sum e_i^2}{n}}$

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3. $P(Y=1) = P(Y^* > 0) = P(a + bx + \epsilon > 0)$
$= P(\epsilon > -a - bx) = 1 - \underset{(CDF)}{F_\epsilon}(-a - bx)$

$P(Y=0) = P(Y^* < 0) = P(a + bx + \epsilon < 0)$
$= P(\epsilon < -a - bx) = F_\epsilon(-a - bx)$

$\mathcal{L}(a, b) = \prod_{i=1}^{n} [P(y_i = 1))^{y_i} (P(y_i = 0))^{1-y_i}$

$\log \mathcal{L}(a,b) = \sum_{i=1}^{n} y_i \log(P(y_i = 1)) + (1 - y_i)$
$* \log(P(y_i = 0))$

So $\log \mathcal{L}(a,b) = \sum y_i \log\left[1 - \frac{1}{1+e^{a+bx_i}}\right]$
$+ (1-y) \log\left[\frac{1}{1+e^{a+bx_i}}\right]$

$\frac{\partial}{\partial a} \log \mathcal{L} = \sum y_i \cdot \frac{1}{1 - \frac{1}{e^{a+bx_i}}} \left(\frac{e^{a+bx_i}}{1+e^{a+bx_i}}\right)$
$+ (1 - y_i) \frac{e^{a+bx_i}}{(1+e^{a+bx_i})} = 0$

$\sum y_i \left[\frac{1+e^{a+bx_i}}{1+e^{a+bx_i}}\right] - \left[\frac{e^{a+bx_i}}{1+e^{a+bx_i}}\right]$

$$\frac{\partial \log \mathcal{L}(a,b)}{\partial a} = \Sigma y_i - \Sigma \frac{e^{a+bx_i}}{1+e^{a+bx_i}} = 0$$

$$\frac{\partial \log \mathcal{L}(a,b)}{\partial b} = \Sigma x_i y_i - \Sigma \frac{e^{a+bx_i}}{1+e^{a+bx_i}} (X_i) = 0$$

$$S_0: \begin{cases} \Sigma y_i - \Sigma \frac{e^{a+bx_i}}{1+e^{a+bx_i}} = 0 \\ \Sigma x_i y_i - \Sigma \frac{e^{a+bx_i}}{1+e^{a+bx_i}} = 0 \end{cases}$$

THERE IS NO CLOSED-FORM SOLUTION
BUT WE CAN APPROXIMATE IN R.